

# Quantum teleprotation with sonic black holes

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## Abstract

We show a new property of sonic black holes. After deriving the metric of a sonic black hole from the Schrödinger equation and quantizing the perturbation fields near the sonic event horizon, we show particles of Hawking radiation can act as a source of entanglement: two-mode squeezed entanglement is produced near the event horizon, which can be used in quantum teleportation. The fidelity of the teleportation is closely related to the temperature of the sonic black holes, but high fidelity seems difficult to reach in our case.

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Recent investigation in string theory indicates that black hole may not actually destroy the information about how they were formed, but instead process it and emit the processed information as part of the Hawking radiation as they evaporate[1-5]. Lloyd suggested that [6], if this is correct, then black holes can act as quantum computers: when a black hole is formed, information is encoded in the initial conditions to be processed by the planckian dynamics at the hole's horizon, and answers can be extracted to the computation by examining the correlation in the Hawking radiation emitted when the hole evaporate. Lloyd's proposal is supported by Horowitz and Maldecena's (HM) theory of "the black hole final state"[7]. HM have proposed a simple model of black hole evaporation by imposing a final boundary condition near the black hole singularity. The annihilation of infalling particles and the collapsed matter can act as a measurement, transferring the information contained in the matter to the outgoing Hawking radiation. Although interesting, HM's conjecture seems fall outside the usual formulation of quantum mechanics and was criticized by several authors[8-11]. Moreover, whether Hawking radiation can transfer information also remains as an enigma to most people.

In 1981, Unruh developed a way of mapping certain aspects of black holes in supersonic flows and pointed out that propagation of sound in a fluid or gas turning supersonic[12], is similar to the the propagation of a scalar field close to a black hole, and thus experimental investigation of the Hawking radiation is possible. From then on, several candidates have been considered for the experimental test of the sonic analog of black holes, which include superfluid helium II[13], atomic Bose-Einstein condensates[14], one-dimensional (1D) transonic flows[15], dielectric[16], wave gravity[17], and 1D Fermi-degenerate noninteracting gas[18]. Similar to classical black holes, sonic black holes also have the structure of ergosphere, trapped regions, apparent horizon, and event horizon[14]. The difference is that sonic black hole has no singularity. In this paper, we will demonstrate quantum teleportation can be realized by using the squeezed vacuum states produced from a sonic black hole's horizon. The fidelity of the teleportation is closely related to the temperature of a sonic black hole.

Since the discussions on the Hawking radiation in sonic black holes have been extended to quantum fluids, we will first derive the metric of a sonic black hole and start from the Schrödinger equation, which can result in the (barotropic, inviscid, and irrotational) hydrodynamics equations[19]. Then we will quantize the fluctuation field and see how quantum teleportation is possible with a sonic black hole. The non-relativistic quantum mechanics can be described by the Schrödinger equation  $[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})]\psi(\vec{x}, t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t)$ , where  $V(\vec{x})$  is an effective potential, and  $\psi(\vec{x}, t)$  might be considered as a macrostate of condensate matter. The Lagrangian corresponding to the Schrödinger equation can be defined as  $L = \frac{i\hbar}{2}(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}) - \frac{\hbar^2}{2m}\nabla\psi^* \cdot$

$\nabla\psi - V(\vec{x})\psi^*\psi$ , which can be further rewritten as

$$L = -\psi^*\psi\left\{\frac{\hbar}{2i}\frac{\partial}{\partial t}\ln\psi/\psi^* + \frac{\hbar^2}{2m}\nabla(\ln\psi^*)\nabla(\ln\psi) + V(\vec{x})\right\}. \quad (1)$$

Comparing Eq.(1) with the Jacobi-Hamilton equation, i.e.

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\vec{x}) = 0, \quad (2)$$

where  $S$  is the action of the whole system, one obtains  $S = S^* = \frac{\hbar}{i}\ln\psi = -\frac{\hbar}{i}\ln\psi^*$ . Assumed  $S = S_r + iS_i$ ,  $\psi$  can be rewritten as

$$\psi(\vec{x}, t) = e^{\frac{i(S_r + iS_i)}{\hbar}} = \rho^{1/2}(\vec{x}, t)e^{\frac{iS_r(\vec{x}, t)}{\hbar}}, \quad (3)$$

where  $\rho = \psi\psi^*$  is the probability density. Substituting Eq.(3) into the Schrödinger equation, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j}(\vec{x}, t) = 0, \quad (4)$$

$$\frac{\partial S_r}{\partial t} + \frac{(\nabla S_r)^2}{2m} + V(\vec{x}) - \frac{\hbar^2}{2m} \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} = 0, \quad (5)$$

where  $\vec{j}(\vec{x}, t) = \frac{\rho}{m}\nabla S_r$  and the last term in Eq.(5) corresponds to the quantum effect without classical correspondence. We may drop out this term in that  $\hbar^2$  is small and  $\rho \leq 1$ . If we set  $\vec{v} = \nabla S_r/m$ , then Eq.(5) can be rewritten as

$$\frac{\partial \vec{v}}{\partial t} + \nabla(\vec{v}^2/2) = -\nabla V(\vec{x})/m, \quad (6)$$

which is the exact equation of irrotational fluid. Defining  $\Phi = S_r/m$  and  $V(\vec{x}) = \phi(\vec{x}) + \int_0^p dp'/\rho(p')$ , we have

$$\frac{\partial \Phi}{\partial t} + \vec{v}^2/2 = \frac{\phi(\vec{x})}{m} - \frac{1}{m} \int_0^p dp'/\rho(p'), \quad (7)$$

where  $p'$  denotes the pressure and  $\phi(\vec{x})$  denotes the external potential. By further defining  $\xi = \ln\rho$  and  $g(\xi) = \int_0^p dp'/\rho(p')$ , we have the new forms of Eq.(4) and Eq.(7)

$$\partial\xi/\partial t + \vec{v} \cdot \nabla\xi + \nabla \cdot \vec{v} = 0, \quad (8)$$

$$\frac{\partial \Phi}{\partial t} + \vec{v}^2/2 - \frac{\phi(\vec{x})}{m} + g(\xi)/m = 0. \quad (9)$$

Linearizing Eq.(8) and Eq.(9) around the assumed background  $(\xi_0, \Phi_0)$ , with

$\xi = \xi_0 + \tilde{\xi}$  and  $\Phi = \Phi_0 + \tilde{\Phi}$ , we obtain

$$\rho_0^{-1}[\partial\rho_0\tilde{\xi}/\partial t + \nabla \cdot (\rho_0\vec{v}\tilde{\xi} + \rho_0\nabla\tilde{\Phi})] = 0, \partial\tilde{\Phi}/\partial t + \vec{v} \cdot \nabla\tilde{\Phi} + \frac{1}{m}g'(\xi_0)\tilde{\xi} = 0, \quad (10)$$

which result in an equation for  $\tilde{\Phi}$ ,

$$\rho_0^{-1} \left\{ \frac{\partial}{\partial t} \left( \frac{m\rho_0}{g'(\xi_0)} \frac{\partial\tilde{\Phi}}{\partial t} + \frac{m\rho_0\vec{v}_0}{g'(\xi_0)} \cdot \nabla\tilde{\Phi} \right) + \nabla \cdot \left( \frac{m\rho_0\vec{v}}{g'(\xi_0)} \frac{\partial\tilde{\Phi}}{\partial t} - \rho_0\nabla\tilde{\Phi} + \vec{v} \frac{m\rho_0\vec{v}}{g'(\xi_0)} \cdot \nabla\tilde{\Phi} \right) \right\} = 0, \quad (11)$$

which describes the propagation of the a massless scalar field in terms of metric. The local speed of sound is defined by  $c^2 \equiv g'(\ln\rho_0)/m$ , which is assumed to be a constant. The metric can be written as  $ds^2 = \rho_0/c[(c^2 - v_0^2)dt^2 + 2\vec{v}_0 \cdot d\vec{x}dt - d\vec{x} \cdot d\vec{x}]$ . Assuming that the background flow is a spherically symmetric, stationary, and convergent flow, we can define a new time coordinate by  $d\tau = dt + \frac{\vec{v}_0 \cdot d\vec{r}}{c^2 - v_0^2}$ . Substituting this back into the line element gives

$$ds^2 = \frac{\rho_0}{c} \left[ (c^2 - v_0^2)d\tau^2 - \frac{c^2}{c^2 - v_0^2}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right] \quad (12)$$

As suggested by Unruh that at  $r = R$ , we assume the background fluid smoothly exceeding the velocity of sound,  $v_0 = -c + \alpha(r - R) + O((r - R)^2)$ . Then the metric can be reexpressed as

$$ds^2 = \frac{\rho_0}{c} \left( 2c\alpha(r - R)d\tau^2 - \frac{cdr^2}{2\alpha(r - R)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right), \quad (13)$$

which has the same form of the Schwarzschild metric. Dropping the angular part of the metric will not change our results in the following discussion.

When we quantize the field  $\tilde{\Phi}$ , we will find the behavior of the normal modes near the sonic horizon implies that this sonic black hole will emit sound waves with a thermal spectrum and particles will be produced near the horizon. The particles created from the vacuum near the horizon are actually the modes of squeezed vacuum states, which are approximate to the EPR (Einstein-Podolsky-Rosen) states and can be used in quantum teleportation. A distant observer from the horizon ( $r \gg R$ ) can be regarded as an observer in Minkowski space. Thus, a massless scalar quantum field  $\tilde{\Phi}$  in the D-dimensional Minkowski space-time can be decomposed in Minkowski modes  $\{U_k(x)\}$ , which goes as

$$\tilde{\Phi} = \sum_k \left[ a_k U_k(x) + a_k^\dagger U_k^*(x) \right], \quad (14)$$

where  $a_k, a_k^\dagger$  are annihilation and creation operators respectively, the boundary conditions  $k_n^i = 2\pi L_i^{-1} n_i$  ( $i = 1, \dots, D$ ) and  $k = (k_1, \vec{k})$ . The Minkowski vacuum can be defined by

$$a_k | 0_M \rangle = 0, \forall k. \quad (15)$$

By solving the Klein-Gordon equation in the coordinates of Eq. (13), the field  $\tilde{\Phi}$  can be expanded in normal modes:

$$\tilde{\Phi} = \sum_{\sigma} \sum_p \left[ b_p^{(\sigma)} u_p^{(\sigma)}(x) + b_p^{(\sigma)\dagger} u_p^{(\sigma)*}(x) \right], \quad (16)$$

where the operators  $b_p^{(\sigma)}$  and  $b_p^{(\sigma)\dagger}$  are assumed to satisfy the usual canonical commutation relations,  $p = (\Omega, \vec{k})$ , and the symbol  $\sigma = \pm$  refers to region I and II respectively, which is separated by the event horizon. By introducing the Unruh modes[20]

$$d_p^{(\sigma)} = \int_{-\infty}^{\infty} d\vec{k} p_{\Omega}^{(\sigma)}(\vec{k}) a_{k_1, \vec{k}}, \quad (17)$$

where  $\{p_{\Omega}^{(\sigma)}(\vec{k})\}$  is the complete set of orthogonal functions. The modes  $b_p^{(\sigma)}$  and  $b_{\tilde{p}}^{(-\sigma)}$  can be well expressed in terms of the Unruh modes by the Bogolubov transformations

$$b_p^{(\sigma)} = [2\sinh(2\omega\pi/\alpha)]^{-\frac{1}{2}} \left[ e^{\omega\pi/\alpha} d_p^{(\sigma)} + e^{-\omega\pi/\alpha} d_{\tilde{p}}^{(-\sigma)\dagger} \right], \quad (18)$$

$$b_{\tilde{p}}^{(-\sigma)} = [2\sinh(2\omega\pi/\alpha)]^{-\frac{1}{2}} \left[ e^{-\omega\pi/\alpha} d_p^{(\sigma)\dagger} + e^{\omega\pi/\alpha} d_{\tilde{p}}^{(-\sigma)} \right], \quad (19)$$

where  $\tilde{p} = (\Omega, -\vec{k})$ . One then obtains[20]

$$| 0 \rangle_M = Z \prod_{\sigma, p} \exp(\tanh r b_p^{(\sigma)\dagger} b_{\tilde{p}}^{(-\sigma)\dagger}) | 0_I \rangle \otimes | 0_{II} \rangle, \quad (20)$$

where  $Z$  is a normalization constant  $Z = \prod_p \cosh^{-2} r$ , where  $r = r(p)$ ,  $\tanh r = e^{-\omega\pi/\alpha}$ , and  $\cosh r = (1 - e^{-2\omega\pi/\alpha})^{-1/2}$ . Assumed

$$\begin{aligned} T^{(+)}(r) &= - \sum_p (b_p^{(+)\dagger} b_p^{(+)} \ln \sinh^2 r - b_p^{(+)} b_p^{(+)\dagger} \ln \cosh^2 r), \\ T^{(-)}(r) &= - \sum_p (b_{\tilde{p}}^{(-)\dagger} b_{\tilde{p}}^{(-)} \ln \sinh^2 r - b_{\tilde{p}}^{(-)} b_{\tilde{p}}^{(-)\dagger} \ln \cosh^2 r), \end{aligned} \quad (21)$$

the following can be easily proved

$$\begin{aligned} e^{-T^{(\sigma)}(r)/2} b_p^{(\sigma)\dagger} e^{-T^{(\sigma)}(r)/2} &= b_{(p)}^{(\sigma)\dagger} \tanh r, \\ e^{-T^{(\sigma)}(r)/2} | 0 \rangle_{I, II} &= \prod_p \cosh^{-1} r | 0_I \rangle \otimes | 0_{II} \rangle. \end{aligned} \quad (22)$$

By using Eqs.(22), one can rewrite Eq.(20) as

$$\begin{aligned}
|0\rangle_M &= \left( e^{-T^{(\sigma)}/2} e^{\sum_p b_p^{(\sigma)\dagger} b_{\tilde{p}}^{(-\sigma)}} e^{T^{(\sigma)}/2} \right) \left( e^{-T^{(\sigma)}/2} |0\rangle_{I,II} \right) \\
&= e^{\sum_p [n_p \ln \sinh r - (1+n_p) \ln \cosh r]} \sum_{n_p=0}^{\infty} |n_p\rangle_I \otimes |n_{\tilde{p}}\rangle_{II} \\
&= \sum_{n_p=0}^{\infty} \prod_p \tanh^{n_p} r \cosh^{-1} r |n_p\rangle_I \otimes |n_{\tilde{p}}\rangle_{II},
\end{aligned} \tag{23}$$

where  $N_p = b_p^{(\sigma)\dagger} b_p^{(\sigma)}$ ,  $b_p^{(\sigma)} b_{\tilde{p}}^{(\sigma)\dagger} = 1 + N_p$ . From Eq.(17), we see that a given Minkowski mode of frequency  $\omega_{\vec{k}}$  is spread over all positive frequencies  $\Omega$ , as a result of the Fourier transform relationship between  $a_{k_1, \vec{k}}$  and  $d_p^{(\sigma)}$  [21]. In fact, only one mode produced by the sonic horizon is enough for our purpose. Thus, one can only consider the mode  $p$  in region I and the mode  $\tilde{p}$  in region II. Therefore, the single mode component of the Minkowski vacuum state, namely the two-mode vacuum squeezed state can be given by

$$|0\rangle_M = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh^n r |n\rangle_I \otimes |n\rangle_{II}. \tag{24}$$

Equation (24) demonstrates that the vacuum appears as an entangled state of Hawking particles with nonlocal EPR type correlations and the vacuum can approach a perfectly entangled state of particle number asymptotically in the limit  $\tanh r \rightarrow 1$ . The reduced density of each mode is a thermal-like state with mean particle number  $\bar{n} = \frac{1}{e^{2\pi\omega/\alpha} - 1}$ , with the temperature  $T = \frac{\hbar\alpha}{2\pi k}$ , which is the Hawking temperature of the sonic black hole. As the above two-mode squeezed vacuum state is produced by the sonic horizon, we can set up the teleportation protocol: suppose one mode is open to local operations and measurements at the sender's location A by observer Alice, while the other mode is open to local operations and measurements in the receiver's location B, by observer Bob. Alice and Bob can communicate with each other via a classical channel. In the protocol, we consider the target mode  $T$ , in an unknown state  $|\varphi\rangle_T$ . The input state of the complete system is

$$\cosh^{-1} r \sum_{k=1}^{\infty} \tanh^k r |\varphi_T\rangle \otimes |k_I\rangle \otimes |k_{II}\rangle \tag{25}$$

The teleportation protocol here require Alice make measurements on T and I by using a unitary operator  $U$ , which can be defined as  $U = S_1 \otimes S_2$ . Suppose the combined system state after Alice's measurement is  $|\Theta\rangle \otimes |X_{II}\rangle$ , where  $|X_{II}\rangle$  is the state of Bob after the teleportation and  $|\Theta\rangle = \gamma \sum_{j=1}^{\infty} |j_T\rangle \otimes |j_A\rangle$ ,  $\gamma$  is the normalization constant and  $|j_T\rangle (j = 1, 2, \dots)$  are the fixed orthonormal bases for the Hilbert space of the target state  $\varphi_T$ . Thus, we have

$$\begin{aligned}
|X_{II}\rangle &= \gamma \cosh^{-1} r \sum_{j=1}^{\infty} \langle j_T | \otimes \langle j_A | (S_1 \otimes S_2) \otimes \sum_{k=1}^{\infty} \tanh^k r |\varphi_T\rangle \otimes |k_I\rangle \otimes |k_{II}\rangle \\
&= \gamma \cosh^{-1} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \tanh^k r \langle j_T | S_1 | \varphi_T \rangle \langle j_A | S_2 | k_I \rangle |k_{II}\rangle
\end{aligned} \tag{26}$$

In terms of the basis components  $X_{IIj} = \langle j_{II} | X_{II} \rangle$ ,  $\varphi_k = \langle k_T | \varphi_T \rangle$ ,  $S_{1jk} = \langle j_T | S_1 | k_T \rangle$  and  $S_{2jk} = \langle j_I | S_2 | k_I \rangle$ , we have

$$X_{IIj} = \gamma \cosh^{-1} r \sum_{l=1}^{\infty} \tanh^l r (S_2^T S_1)_{jl} \varphi_{Tl}, \tag{27}$$

where  $S^T$  denotes matrix transpose of  $S$ . After Alice's measurements, she can inform Bob what the unitary transform  $S_2^T S_1$  is via the classical channel. The original work on teleportation with squeezed vacuum state has been done by Milburn and Braunstein (MB) in 1999 [22,23]. They pointed out that the entanglement of Eq.(24) can be viewed in two ways: as an entanglement between quadrature phases or as an entanglement between number and phase in the two modes[22]. In case of teleportation using number and phase measurements, the modes obtained by Bob can be expressed as[22]

$$|X^{(\pm k)}\rangle = \frac{(1 - \tanh^2 r)^{1/2}}{\sqrt{P_{\pm}(k)}} \sum_{n=2k,0}^{\infty} \tanh^{n \pm 2k} r c_n |n \pm 2k\rangle_{II}, \tag{28}$$

where  $P_{\pm}(k) = (1 - \tanh^2 r) \sum_{n=0,2k}^{\infty} \lambda^{2n} |c_{n \pm 2k}|^2$ . MB's results show that when the mean particle number in the entanglement resource is significantly greater than in the target state, the teleportation has high fidelity. In particular, if the target state is a coherent state with amplitude  $\varsigma$ , the fidelity for zero particle-number difference measurement is[22]

$$F(0) = e^{-|\varsigma|^2(1 - \tanh r)^2}. \tag{29}$$

We see that if  $\tanh r \rightarrow 1$  and the photon-number is zero, then the teleportation is perfect, which infers that  $\alpha \rightarrow \infty$ . However, considered the velocity can not change greatly when liquid pass the horizon,  $\alpha \rightarrow \infty$  is not permitted. Therefore, perfect teleportation seems impossible by using the two-mode squeezed vacuum state from a sonic black hole horizon.

In summary, we have investigate the possibility of using particles of Hawking radiation from sonic black hole horizon as a source of quantum teleportation. The fidelity of teleportation is shown rely on the the Hawking temperature in that  $T \propto \alpha$ . Considered the recent report on Hawking temperature of sonic black holes( $\sim 200nK$ )[18], perfect teleportation seems impossible to reach by using sonic black holes.

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